On the orbits of Kaprekar's transformations by discussing the numerical results

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Abstract. The paper is devoted to the discussion on the orbits of Kaprekar's transformations generated and analyzed only on the basis of selected numerical results. We succeeded in discovering many intriguing facts, for instance the fact that the number of elements of the orbits of the first fifty Kaprekar's transformations "stabilizes" quickly and is not greater than seven! Moreover, "almost" all the numerical orbits obtained by us can be expanded in the infinite sequences of the orbits of the respective Kaprekar's transformations with the subscripts increasing at arithmetic rate. We have also discovered only one single "anomaly", that is the orbit which we were able to expand only in a finite "extinguishing" sequence of orbits of the seven successive Kaprekar's transformations.
From the numerical side we used two algorithms, one is based on the application of the prime divisors of the investigated numbers and the other one is grounded on the extraction of the maximal subset with respect to the inclusion relation, invariant with regard to the discussed Kaprekar's transformation.

Keywords: Kaprekar's transformation, orbits of Kaprekar's transformation, divisibility rules.

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1. Introduction

The paper is already the fourth one in the cycle of papers written by the authors and devoted to the Kaprekar's transformations (see [7, 8, 6]). The topic of Kaprekar's transformations has inspired many authors which can be seen for instance by investigating the references included in our papers. It is not a secret that our actions, as researchers, were mostly stimulated by some numerical aspects connected especially with the number and the forms of orbits of the Kaprekar's transformations.

Let us fix $n \in \mathbb{N}$, $n \geq 2$. Let $\alpha \in \mathbb{N}$ be any n-digit number in its decimal expansion, the digits of which are ordered in the following nondecreasing sequence

$$0 \leq a_1 \leq a_2 \leq \ldots \leq a_n \leq 9.$$

Let us also assume that at least two digits are different, that is the condition $a_1 \neq a_n$ is satisfied. We take

$$T_n(\alpha) := \sum_{k=1}^{n} (a_k - a_{n-k+1}) 10^{k-1} = a_n a_{n-1} \ldots a_1 - a_1 a_2 \ldots a_n. \tag{1}$$

The map T_n is called the n-th Kaprekar's transformation, we will also use the terms "operator" T_n and "mapping" T_n.

Let us notice that in the decimal expansion of the number $T_n(\alpha)$ at least two digits are then different, additionally $T_n(\alpha) < 10^n$, and finally, by completing, if necessary, the number obtained according to formula (1) with the appropriate number of zeros, we assume also that $10^{n-1} - 1 \leq T_n(\alpha)$. The reason for such an action is the following fact

$$T_n(a_1 a_2 \ldots a_n) = T_n(a_{\sigma(1)} a_{\sigma(2)} \ldots a_{\sigma(n)})$$

for any permutation $\sigma \in S_n$, where, as usually, S_n denotes the set of all elements of the symmetric group of order n.

For example, we have

$$\begin{gathered} T_3(323) = 332 - 233 = 99 = 099, \\ T_4(0999) = T_4(9099) = T_4(9909) = T_4(9990) = 8991. \end{gathered} \tag{2}$$

Let us introduce the following notations

$$a(k\times) := a\left(10^{k-1} + 10^{k-2} + \ldots + 1\right) = \underbrace{a \ldots a}_{k\, times}$$

for any $a \in \{0, 1, 2, \ldots, 9\}$ and $k \in \mathbb{N}$ (if $k \in \mathbb{Z}$, $k \leq 0$, then we set $a(k\times) := \emptyset$) and

$$\mathbb{N}_k^{cph} := \left\{ n \in \mathbb{N} \colon 10^{k-1} - 1 \leq n < 10^k \wedge n \neq a(k\times),\ \text{where } a \in \{1, 2, \ldots, 9\} \right\},$$

for $k = 2, 3, \ldots$, that is $\mathbb{N}_k^{cph}$ denotes the set composed of the number $10^{k-1} - 1 := 09((k-1)\times)$ and these natural numbers, the decimal expansion of which contains k digits and, simultaneously, at least two different digits.

Remark 1.1. Identifying the notation $09(k\times)$ with the $(k+1)$-element sequence of the respective numbers, similarly as the remaining numbers from the set $\mathbb{N}_{k+1}^{cph}$ for every $k = 1, 2, 3, \ldots$, we can write that

$$\mathbb{N}_k^{cph} \cap \mathbb{N}_{k+1}^{cph} = \emptyset, \ k \in \mathbb{N}.$$

By using the notations introduced above we can additionally formulate the following theorem referring to example (2) (see [6]):

Theorem 1.2. *If $n \in \mathbb{N}_k^{cph}$, then $T_k(n) \geq 9((k-1)\times)$ for every $k = 2, 3, \ldots$*

Corollary 1.3. *We have*

$$T_k \colon \mathbb{N}_k^{cph} \to \mathbb{N}_k^{cph}$$

for every $k = 2, 3, \ldots$ Moreover

$$T_k\left(09(k\times)\right) = 89((k-1)\times)1,$$

$$T_k\left(a(k\times)(a-1)\right) = 09(k\times)$$

for every $a = 2, 3, \ldots, 9$ and $k = 2, 3, \ldots$

Remark 1.4. The reason for introducing the set $\mathbb{N}_k^{cph}$ in this paper was to eliminate from the discussion the trivial fixed point (i.e. the zero number) of the transformations T_k. Let us notice that in papers [7, 8] the trivial fixed point of the transformations T_k is allowed. In this case we had a more general definition of the Kaprekar's transformation

$$T_k \colon \{0\} \cup \left\{\alpha \in \mathbb{N} \colon 10^{k-1} - 1 \leq \alpha < 10^k\right\} \to \{0\} \cup \mathbb{N}_k^{cph},$$

where $k \geq 2$. Certainly $T_k(0) = 0$. Let us also notice that we have then (by definition (1), without the assumption that $a_1 \neq a_n$):

$$T_k\left(\{0\} \cup \left\{\alpha \in \mathbb{N} \colon 10^{k-1} - 1 \leq \alpha < 10^k\right\}\right) = \{0\} \cup \mathbb{N}_k^{cph}.$$

2. Orbits of Kaprekar's transformations T_n for $n \leq 50$

We begin with some general conclusions concerning the orbits of Kaprekar's transformations drawn just by observing the forms of orbits of the Kaprekar's transformations obtained with the aid of computer calculations.

Thus, we have observed that the odd Kaprekar's transformations (that is, by definition, the mappings T_n with an odd subscript n) starting from T_{13} till T_{49} possess only 1-, 2- , 3- and 5-element orbits. As a quite intriguing fact, let us notice that each mapping $T_{13+2n}, n \in \mathbb{N}_0$ possesses the following 2-element orbit

$$\left(873((n+2)\times)209876((n+2)\times)22, \quad 966543(n\times)296((n+1)\times)54331\right).$$

Verification. We have: $T_{2n+13}(873(n+2\times)209876(n+2\times)22)=$

$$\begin{array}{r} = 988776(\,n+2\times\,)3(\,n+2\times\,)2220 \\ -\ 02223(\,n+2\times\,)6(\,n+2\times\,)77889 \\ \hline 966543(\ \ n\times\ \ )296(n+1\times)54331, \end{array}$$

and $T_{2n+13}(966543(n\times)296(n+1\times)54331) =$

$$\begin{array}{r} = 99666(n+1\times)5544333(\ \ n\times\ \ )21 \\ -12333(\ \ n\times\ \ )4455666(n+1\times)99 \\ \hline 873(\,n+2\times\,)209876(\,n+2\times\,)22. \end{array}$$

Remark 2.1. To the contrast the numbers 87**3**20987**6**22 and 96653954331 belong to the orbit of length 8 of T_{11} and we have $T_{11}(87320987622) = 96653954331$.

Moreover, except the following cases: T_{23}, T_{25}, T_{41} and T_{43} possessing two 2-element orbits and the transformations T_{37} and T_{39} possessing three 2-element orbits, all the others odd Kaprekar's transformations have exactly one 2-element orbit.

Whereas the even Kaprekar's transformations (that is the mappings T_n with an even subscript n) starting from T_2 possess only 1-, 2-, 3-, 5- and 7-element orbits. The 5-element orbits are possessed only by the following transformations: T_{20} – one such orbit, T_{22} – three such orbits, T_{24} – five such orbits and T_{2n}, $n \in \{13, \ldots, 18\}$ – six such orbits, next T_{38} – seven such orbits, T_{40} – nine such orbits, T_{42} – eleven such orbits and, at last, T_{2n}, $n \in \{22, \ldots, 25\}$ — twelve such orbits.

Remark 2.2. We also have

$$T_2(\mathbb{N}_2^{cph}) = \{A(9-A)\colon\, A = 0, 1, \ldots, 8\}$$

and this set is the only orbit of T_2.

Additionally, the transformations T_{2n+1} possess many other interesting properties:

— the number of 3-element orbits increases with the increasing values of subscript n (from $n = 9$ till $n = 24$) starting from one orbit, through 4, 10, 21, 39, 66, 105, 159, 231, 326, 449, 605, 801, 1044, 1341 orbits and finally till 1701 orbits, whereas the number of fixed points increases also with the increasing values of subscript n (from $n = 8$ till $n = 24$) starting from two points, through 3, 5, 7, 8, 12, 14, 17, 21, 25, 31, 36, 43, 50, 59, 67 fixed points and finally till 76 fixed points. To the contrast, the numbers of 5-element orbits form the nondecreasing sequence: T_{11} – one such orbit, T_{13} – three such orbits, T_{15} – five such orbits, T_{17}–T_{27} – each one possesses six such orbits, T_{29} – seven such orbits, T_{31} – nine such orbits, T_{33} – eleven such orbits, T_{35}–T_{45} – each one possesses twelve such orbits, T_{47} – thirteen such orbits and, at last, T_{49} – fifteen such orbits;

— for $n = 4$ and $n = 7$ we have two fixed points, whereas for $n = 5$ and $n = 6$ we have one fixed point.

For the transformations T_{2n} where $n \in \{4, 5, \ldots, 25\}$, the following facts hold true:

— the sequence of the numbers of 3-element orbits is increasing starting from one orbit, through 4, 10, 20, 36, 60, 94, 141, 204, 286, 392, 527, 696, 906, 1164, 1477, 1854, 2304, 2836, 3462, 4194 orbits and finally till 5044 orbits;

— the sequence of the numbers of fixed points is increasing starting from three fixed points, through 4, 6, 7, 9, 12, 14, 17, 21, 25, 30, 36, 43, 49, 58, 66, 75, 86, 97, 110, 122 fixed points and finally till 137 fixed points;

— the 2-element orbits exist only for $n = 8$, 15, 16, 17, 22, 24, 25, more precisely, there are two 2-element orbits only for $n = 15$ and 24, whereas for $n = 8$, 16, 17, 25 we have only one such orbit. For $n = 22$ we have three 2-element orbits.

Basing on the observation of the forms of these orbits, the following theorems have been deduced. But for clarity let us formulate first the following important editorial remark.

Remark 2.3. From the following theorem on, and henceforward as well, with regard to the length of notation of the respective decimal expansions of the given numbers, we break this notation if it is needed (that is if we have too many digits in this expansion we may transfer them into a new line). For example, instead of notation 12345 in some decimal expansion we may write

$$\begin{array}{l} 12... \\ \qquad ...345 \end{array}$$

Theorem 2.4. *Each transformation T_{16n}, $n = 1, 2, \ldots$, has the following 2-element orbit*

$$\Big(8(n\times)7(n\times)6(n\times)4((2n)\times)2((2n-1)\times)1\ldots$$

$$\ldots 9((2n)\times)7((2n)\times)5((2n)\times)3(n\times)2(n\times)1((n-1)\times)2,$$

$$8(n\times)7(n\times)6(n\times)5(n\times)4(n\times)3(n\times)2((n-1)\times)1\ldots$$

$$\ldots 9((2n)\times)7(n\times)6(n\times)5(n\times)4(n\times)3(n\times)2(n\times)1((n-1)\times)2\Big).$$

Moreover, the transformations $T_{14n+9}, n = 1, 2, \ldots$, have the following 2-element orbit

$$\Big(87((2n)\times)65((n+1)\times)4(n\times)3((n+1)\times)2((n-1)\times)1\ldots$$

$$\ldots 9((2n+1)\times)7(n\times)6((n+1)\times)5(n\times)4((n+1)\times)32((2n+1)\times),$$

$$87((2n)\times)65((n-1)\times)4((n+2)\times)3((n-1)\times)2((n+1)\times)1\ldots$$

$$\ldots 9((2n+1)\times)7((n+2)\times)6((n-1)\times)5((n+2)\times)4((n-1)\times)32((2n+1)\times\Big),$$

and the transformations T_{14n+11}, for every $n = 1, 2, \ldots$, possess a 2-element orbit of the form

$$\Big(887((2n-1)\times)6654((2n)\times)32((2n-1)\times)1\ldots$$

$$\ldots 9((2n+1)\times)7((2n)\times)65((2n)\times)4332((2n-1)\times)12,$$

$$887((2n-1)\times)665((2n-2)\times)4443((2n-2)\times)221\ldots$$

$$\ldots 9((2n+1)\times)7776((2n-2)\times)5554((2n-2)\times)3332((2n-1)\times)12\Big).$$

Proof. For T_{14n+9} we have

$$\begin{array}{r}
T_{14n+9}(87((2n)\times)65((n+1)\times)\ldots 32((2n+1)\times)) = \\
= 9\underbrace{9\ldots9}_{2n}8\underbrace{7\ \ldots\ 7}_{3n}\underbrace{6\ldots6}_{n+2}\underbrace{5\ldots5}_{2n+1}\underbrace{4\ \ldots\ 4}_{2n+1}\underbrace{3\ldots3}_{n+2}\underbrace{2\ \ldots\ 2}_{3n}1 \\
-\ 1\underbrace{2\ \ldots\ 2}_{3n}\underbrace{3\ldots3}_{n+2}\underbrace{4\ \ldots\ 4}_{2n+1}\underbrace{5\ldots5}_{2n+1}\underbrace{6\ldots6}_{n+2}\underbrace{7\ \ldots\ 7}_{3n}8\underbrace{9\ldots9}_{2n+1} \\
\hline
8\underbrace{7\ldots7}_{2n}6\underbrace{5\ldots5}_{n-1}\underbrace{4\ldots4}_{n+2}\underbrace{3\ldots3}_{n-1}\underbrace{2\ldots2}_{n+1}1\underbrace{9\ldots9}_{2n+1}\underbrace{7\ldots7}_{n+2}\underbrace{6\ldots6}_{n-1}\underbrace{5\ldots5}_{n+2}\underbrace{4\ldots4}_{n-1}3\underbrace{2\ldots2}_{2n+1}
\end{array}$$

and

$$\begin{array}{r}
T_{14n+9}(87((2n)\times)65((n-1)\times)\ldots 4((n-1)\times)32((2n+1)\times)) = \\
= 9\underbrace{9\ldots9}_{2n}8\underbrace{7\ \ldots\ 7}_{3n+2}\underbrace{6\ldots6}_{n}\underbrace{5\ldots5}_{2n+1}\underbrace{4\ \ldots\ 4}_{2n+1}\underbrace{3\ldots3}_{n}\underbrace{2\ \ldots\ 2}_{3n+2}1 \\
-\ 1\underbrace{2\ \ldots\ 2}_{3n+2}\underbrace{3\ldots3}_{n}\underbrace{4\ \ldots\ 4}_{2n+1}\underbrace{5\ldots5}_{2n+1}\underbrace{6\ldots6}_{n}\underbrace{7\ \ldots\ 7}_{3n+2}8\underbrace{9\ldots9}_{2n+1} \\
\hline
8\underbrace{7\ldots7}_{2n}6\underbrace{5\ldots5}_{n+1}\underbrace{4\ldots4}_{n}\underbrace{3\ldots3}_{n+1}\underbrace{2\ldots2}_{n-1}1\underbrace{9\ldots9}_{2n+1}\underbrace{7\ldots7}_{n}\underbrace{6\ldots6}_{n+1}\underbrace{5\ldots5}_{n}\underbrace{4\ldots4}_{n+1}3\underbrace{2\ldots2}_{n+1}
\end{array}$$

For T_{14n+11} we have

$$\begin{array}{r}
T_{14n+11}(887((2n-1)\times)6654((2n)\times)\ldots 12) = \\
= 9\underbrace{9\ldots9}_{2n+1}88\underbrace{7\ \ldots\ 7}_{4n-1}666\underbrace{5\ldots5}_{2n+1}\underbrace{4\ \ldots\ 4}_{2n+1}333\underbrace{2\ \ldots\ 2}_{4n-1}11 \\
-\ 11\underbrace{2\ \ldots\ 2}_{4n-1}333\underbrace{4\ldots4}_{2n+1}\underbrace{5\ldots5}_{2n+1}666\underbrace{7\ \ldots\ 7}_{4n-1}88\underbrace{9\ \ldots\ 9}_{2n+1} \\
\hline
88\underbrace{7\ldots7}_{2n-1}66\underbrace{5\ldots5}_{2n-2}444\underbrace{3\ldots3}_{2n-2}221\underbrace{9\ldots9}_{2n+1}777\underbrace{6\ldots6}_{2n-2}555\underbrace{4\ldots4}_{2n-2}33\underbrace{2\ldots2}_{2n-1}12
\end{array}$$

and

$$\begin{array}{r}
T_{14n+11}(887((2n-1)\times)665((2n-2)\times)\ldots 12) = \\
9\underbrace{9\ldots9}_{2n+1}88\underbrace{7\ \ldots\ 7}_{2n+2}\underbrace{6\ldots6}_{2n}\underbrace{5\ldots5}_{2n+1}\underbrace{4\ \ldots\ 4}_{2n+1}\underbrace{3\ldots3}_{2n}\underbrace{2\ \ldots\ 2}_{2n+2}11 \\
-\ 11\underbrace{2\ \ldots\ 2}_{2n+2}\underbrace{3\ldots3}_{2n}\underbrace{4\ \ldots\ 4}_{2n+1}\underbrace{5\ldots5}_{2n+1}\underbrace{6\ldots6}_{2n}\underbrace{7\ \ldots\ 7}_{2n+2}88\underbrace{9\ldots9}_{2n+1} \\
\hline
88\underbrace{7\ldots7}_{2n-1}665\underbrace{4\ldots4}_{2n}3\underbrace{2\ldots2}_{2n-1}1\underbrace{9\ldots9}_{2n}\underbrace{7\ldots7}_{2n}6\underbrace{5\ldots5}_{2n}433\underbrace{2\ldots2}_{2n-1}12
\end{array}$$

$\square$

Theorem 2.5. *Each transformation T_{14n+2}, $n = 1, 2, \ldots$ possesses the following two 2-element orbits*

$$\Big(87((2n-1)\times)64((2n)\times)2((2n-1)\times)19((2n)\times)7((2n)\times)5((2n)\times)32((2n)\times),$$

$$87((2n\text{–}1)\times)65((2n\text{–}1)\times)43((2n\text{–}1)\times)19((2n)\times)76((2n\text{–}1)\times)54((2n\text{–}1)\times)32((2n)\times)\Big);$$

$\Big(87((2n-1)\times)654((2n-1)\times)32((2n-2)\times)19((2n)\times)7((2n-1)\times)65((2n-1)\times)432((2n)\times),$

$87((2n-1)\times)65((2n-2)\times)443((2n-2)\times)219((2n)\times)776((2n-2)\times)554((2n-2)\times)32((2n)\times)\Big).$

Proof. A simple verification – a similar to that in the proof of Theorem 2.4 – is omitted here. □

Theorem 2.6. *Each transformation* T_{18n-2}, $n = 1, 2, \ldots$, *has the following 2-element orbit*

$$\Big(8((2n-1)\times)76((2n-1)\times)54((2n-1)\times)32((2n-2)\times)1\ldots$$
$$\ldots 9((2n)\times)9((2n-1)\times)65((2n-1)\times)43((2n-1)\times)21((2n-2)\times)2,$$
$$8((2n-1)\times)76((2n-1)\times)4((2n)\times)2((2n-1)\times)1\ldots$$
$$\ldots 9((2n)\times)7((2n)\times)5((2n)\times)3((2n-1)\times)21((2n-2)\times)2\Big).$$

For $n = 1$ *and* $n = 2$ *these are the single 2-element orbits of operator* T_{18n-2}.

Proof. A simple verification – a similar to that in the proof of Theorem 2.4 – is omitted here. □

Remark 2.7. Let us notice that apart from the 2-element orbits described only by means of "formulae" from the three theorems presented above each transformation T_{44} and T_{48} possesses additionally one more 2-element orbit.

2.1. Singularity

Every odd Kaprekar's transformation $T_{37}-T_{43}$ possesses the 2-element orbit. These orbits form together a short regular sequence (that is these orbits can be described by one "analytical" formula, like the formula given below) which finishes with a fixed point in case of T_{45} and which vanishes in case of T_{35}. More precisely, the transformations T_{35+2k} have the orbit

$$\Big(8(k\times)7((5-k)\times)6(k\times)4(5\times)2(4\times)1...$$
$$...9(5\times)7(5\times)5(5\times)3(k\times)2((5-k)\times)1((k-1)\times)2,$$
$$8(k\times)7((5-k)\times)6(k\times)5((5-k)\times)4(k\times)3((5-k)\times)2((k-1)\times)1...$$
$$...9(5\times)7(k\times)6((5-k)\times)5(k\times)4((5-k)\times)3(k\times)2((5-k)\times)1((k-1)\times)2\Big)$$

for each $k = 1, \ldots, 5$.

For $k = 5$ we get the following fixed point of T_{45}:

$$8(5\times)6(5\times)4(5\times)2(4\times)19(5\times)7(5\times)5(5\times)3(5\times)1(4\times)2.$$

It is the only one unusual feature of the Kaprekar's transformations noticed by us till now (on the basis of numerically determined results). Mostly the orbits, noticed by us, "developed" into the infinite sequence of orbits, instead of being "cut" after a finite number of steps.

2.2. One more singularity

We have also observed that each Kaprekar's transformation T_{2n} for $n \geq 3$ possesses the following 7-element orbit

$$\begin{aligned}\Big(&43(k\times)20876(k\times)6, && 853(k\times)176(k\times)42,\\ &753(k\times)086(k\times)43, && 843(k\times)086(k\times)52,\\ &863(k\times)086(k\times)32, && 863(k\times)266(k\times)32,\\ & && 643(k\times)266(k\times)54\Big),\end{aligned}$$

where $k := n - 3$.

For $n \in \{3, 4, \ldots, 25\}$ this is the single 7-element orbit of the transformation T_{2n}.

However we are not sure whether the singularity of 7-element orbit is a common property of all the transformations T_{2n}, $n \in \mathbb{N}$.

2.3. 5-element orbits of Kaprekar's transformations

We begin with the description of 5-element orbits of the even Kaprekar's transformations.

The operators T_{20+2n}, $n = 0, 1, 2, \ldots$, have the following 5-element orbit

$$\begin{gathered}\Big(98666443(n\times)2199776(n\times)5533311,\\ 8886443((n+1)\times)2199776((n+1)\times)553112,\\ 88766443(n\times)2199776(n\times)5533212,\ 88665443(n\times)2199776(n\times)5543312,\\ 8866443(n\times)220998776(n\times)553312\Big).\end{gathered}$$

The operators T_{22+2n}, $n = 0, 1, 2, \ldots$, possess additionally the following 5-element orbits

$$\Big(987666443(n\times)2199776(n\times)55333211,$$
$$88865443((n+1)\times)2199775((n+1)\times)5543112,$$
$$88766443(n\times)220998776(n\times)5533212,$$
$$986665443(n\times)2199776(n\times)55433311,$$
$$8886443((n+1)\times)220998776((n+1)\times)553112\Big)$$

and

$$\Big(98666443(n\times)220998776(n\times)5533311,$$
$$98866443((n+1)\times)2199776((n+1)\times)5533111,$$
$$88876443((n+1)\times)2199776((n+1)\times)5532112,$$
$$887665443(n\times)2199776(n\times)55433212,$$
$$88665443(n\times)220998776(n\times)5543312\Big).$$

The operators T_{24+2n}, $n = 0, 1, 2, \ldots$, possess also some other 2-element orbits, the description of which will be omitted here with regard to the reasonable length of the current paper.

Finally, the operators T_{26+2n}, $n = 0, 1, 2, \ldots$, have one more, the sixth one, 5-element orbit

$$\Big(9887665443((n+1)\times)2199776((n+1)\times)554332111,$$
$$888765443((n+1)\times)220998776((n+1)\times)554332111,$$
$$9876665443(n\times)220998776(n\times)554333211,$$
$$988665443((n+1)\times)220998776((n+1)\times)55433111,$$
$$988766443((n+1)\times)220998776((n+1)\times)55332111\Big).$$

Let us notice that the operators T_{2n}, $13 \le n \le 18$, possess only six presented above 5-element orbits.

Only operator T_{38} has seven 5-element orbits, more precisely, the operator T_{38+2n}, for $n = 0, 1, 2, \ldots$, apart from the six described above 5-element orbits, possesses one more, the seventh one, 5-element orbit

$$\Big(88886666544443(n\times)2221999977776(n\times)5555433331112,$$
$$8888666644443(n\times)222209999877776(n\times)555533331112,$$
$$98886666644443(n\times)2221999977776(n\times)5555333331111,$$
$$88888666644443((n+1)\times)2221999977776((n+1)\times)555533311112,$$
$$88887666644443(n\times)2221999977776(n\times)5555333321112\Big).$$

Next, starting from T_{40} there appear two more 5-element orbits, more precisely, the operator T_{40+2n}, for $n = 0, 1, 2, \ldots$, possesses the following two additional 5-element orbits

$$\Big(98888666644443((n+1)\times)2221999977776((n+1)\times)5555333311111,$$
$$88888766644443((n+1)\times)32221999977776((n+1)\times)5555333211112,$$
$$888876666544443(n\times)2221999977776(n\times)55554333321112,$$
$$88886666544443(n\times)222209999877776(n\times)5555433331112,$$
$$98886666644443(n\times)222209999877776(n\times)5555333331111\Big)$$

and

$$\Big(988866666544443(n\times)2221999977776(n\times)55554333331111,$$
$$8888866644443((n+1)\times)222209999877776((n+1)\times)555533311112,$$
$$988876666644443(n\times)2221999977776(n\times)55553333321111,$$
$$88888666544443((n+1)\times)2221999977776((n+1)\times)5555433311112,$$
$$888876666644443(n\times)222209999877776(n\times)5555333321112\Big).$$

Furthermore, the operator T_{42+2n}, $n = 0, 1, 2, \ldots$, possesses the next two 5-element orbits, whereas T_{44+2n}, $n = 0, 1, 2, \ldots$, has one more such 5-element orbit. Precise description of these additional orbits will be also omitted here. The transformation T_{46+2n}, for $n \leq 2$, does not have any other new 5-element orbits.

Let us also give some examples of other 5-element orbits for the odd Kaprekar's transformations:

— the transformation T_{29+2n} has the orbit

$$\Big(98866664443(n\times)2219997776(n\times)5553333111,$$
$$8888664443((n+1)\times)2219997776((n+1)\times)555331112,$$
$$88876664443(n\times)2219997776(n\times)5553332112,$$
$$88866654443(n\times)2219997776(n\times)5554333112,$$
$$8886664443(n\times)222099987776(n\times)555333112\Big)$$

for every $n = 0, 1, 2, \ldots$;

— the transformation T_{11+2n} has the orbit

$$\Big(96643(n\times)1976(n\times)5331,\ 8843((n+1)\times)1976((n+1)\times)512,$$
$$87643(n\times)1976(n\times)5322,\ 86543(n\times)1976(n\times)5432,$$
$$8643(n\times)209876(n\times)532\Big)$$

for every $n = 0, 1, 2, \ldots$;

— whereas the transformation T_{13+2n} has the orbit

$$\Big(966543(n\times)1976(n\times)54331, 8843((n+1)\times)209876((n+1)\times)512,$$
$$976643(n\times)1976(n\times)53321, 88643((n+1)\times)1976((n+1)\times)5412,$$
$$87643(n\times)209876(n\times)5322\Big)$$

for every $n = 0, 1, 2, \ldots$

3. Singularities of the transformations T_{2n+1}, $n = 1, 2, \ldots, 5$

Singularities of these transformations will be referred to the numbers of the possessed orbits and all the presented facts will be compared with the number and forms of the possessed orbits of the first fifty transformations T_n. Thus

— transformation T_5 has one 2-element orbit and two 4-element orbits; no other odd Kaprekar's transformation possesses the 4-element orbit;

— transformation T_7 has one 8-element orbit;

— transformation T_9 has two fixed points and one 14-element orbit; no other odd Kaprekar's transformation possesses the 14-element orbit;

— transformation T_{11} has one 1-element, one 5-element and one 8-element orbits; only T_7 and T_{11} possess the 8-element orbits!

In our opinion, the above singularities result from the too small number of digits in the decimal expansion of n in the transformation T_n with respect to the way of acting of this transformation (see formula (1)). We also think that these singularities may appear for infinitely many Kaprekar's transformations T_n. And this is, from now on, our basic conjecture!

The above singularities lead immediately to the connotation with some other, more well-known, singularities existing in the number theory, for instance with the divisibility rules (see [17], [14, pp. 17–21], [3, chap. 7]). At least few of these facts we would like to share with our readers. Using the opportunity let us present some original rules of divisibility by 7 and 17 (the latter ones seem to be completely original)!

Rules of divisibility by 7:

let us set a number $k \in \mathbb{N}$ and its decimal expansion $a_n a_{n-1} \ldots a_2 a_1 a_0$.

— typical rule: $7|k$ if and only if the number

$$(a_2 a_1 a_0) - (a_5 a_4 a_3) + (a_8 a_7 a_6) - \ldots$$

is divisible by 7;

— untypical rule: $7|k$ if and only if the number

$$(a_n \ldots a_2 a_1) + 5a_0$$

is divisible by 7;

— another untypical rule: $7|k$ if and only if the number

$$(a_0 + 3a_1 + 2a_2) - (a_3 + 3a_4 + 2a_5) + (a_6 + 3a_7 + 2a_8) - \dots$$

is divisible by 7.

The above rule works also in the ternary numerical system, whereas in the quinary numerical system we have $7|k$ if and only if the number

$$(a_0 - 2a_1 - 3a_2) - (a_3 - 2a_4 - 3a_5) + (a_6 - 2a_7 - 3a_8) - \dots$$

is divisible by 7.

The untypical rules of divisibility by 7 are better if, for example, there is a need of verifying whether the number 1234567 is not divisible by 7.

Rule of divisibility by 17: $17|k$ if and only if the number

$$(a_1a_0 - 2a_3a_2 + 4a_5a_4 - 8a_7a_6) + (a_9a_8 - \dots$$

is divisible by 17.

Taking into account the 3-element segments, or even the 4-element segments, does not generate so effective rules of divisibility by 17 (however one can argue in case of the 4-element segments), for example $17|k$ if and only if the number (we have here the alternating sequence of signs between the successive brackets):

$$\begin{aligned}(a_2a_1a_0 - 3a_5a_4a_3 - 8a_8a_7a_6 + 7a_{11}q_{10}a_9 - 4a_{14}a_{13}a_{12} +\\ - 5a_{17}a_{16}a_{15} - 2a_{20}a_{19}a_{18} + 6a_{23}a_{22}a_{21}) +\\ - (a_{26}a_{25}a_{24} - 3a_{29}a_{28}a_{27} - \text{ and so on }) + \dots\end{aligned}$$

is divisible by 17 or, respectively, the number

$$(a_3a_2a_1a_0 + 4a_7a_6a_5a_4) + (a_{11}a_{10}a_9a_8 + 4a_{15}a_{14}a_{13}a_{12}) + \dots$$

is divisible by 17.

4. Descriptions of numerical algorithms

Computations needed for this paper – connected above all with determination of the orbits of Kaprekar's transformation – have been executed by applying two essentially differing algorithms which we intend to describe in this section. The first algorithm for determination of the orbits is based on the observation that, in case of the Kaprekar's transformation, the order of digits in the decimal expansion of a given number is not important since, let us recall, $T_n(a_{\sigma(1)}a_{\sigma(2)}, \dots, a_{\sigma(n)}) = T_n(a_1a_2 \dots a_n)$ for each permutation $\sigma \in S_n$ and each natural number possessing the n-digit decimal expansion $a_1a_2 \dots a_n$. By this, for the n-digit numbers it is enough to determine all the n-element multisets, the elements of which are the digits. It reduces significantly the calculations because one n-element multiset represents $\frac{n!}{k_0!k_1!\dots k_9!}$ of n-digit numbers, where k_i denotes the number of occurrences of digit i, for $i = 0, 1, ..., 9$, in the decimal description of the given n-digit number.

We should mention in this moment how to generate all the n-element multisets. The procedure is as follows:

- As the first group of multisets we obtain nine 1-element sets
$$\{1\}, \{2\}, \ldots, \{9\}.$$
- To create all the possible 2-element multisets we add to each 1-element set an element greater than or equal to the one contained already in this set. For example, for $\{1\}$ we get $\{1,1\}, \{1,2\}, \{1,3\}, \{1,4\}, \ldots \{1,9\}$.
- We repeat this scheme as long as we obtain a multiset with a proper number of elements. In general, the extension of each multiset can be described in the following way
$$\underbrace{\{c_1, \ldots, c_k\}}_{\text{given multiset}} \longrightarrow \{c_1, \ldots, c_k, c_k\}, \{c_1, \ldots, c_k, c_k + 1\}, \ldots, \{c_1, \ldots, c_k, 9\}$$
where $1 \leq c_1 \leq c_2 \leq \ldots \leq c_k \leq 9$.
- The extensions of $\{0\}$ are considered separately. In this case we take only at least the 2-element multisets containing at least one non-zero element.

We assume that each multiset is represented as the 10-element table in which the i-th element is equal to the number of occurrences of the digit i in the given multiset.

By counting the amount of numbers represented by the given multiset we should to calculate the value of $\frac{n!}{k_0! \ldots k_9!}$. In the numerator of this formula a large number appears which exceeds over the range of integer types. Therefore we use the fact that each natural number, greater than one, can be presented as the product of prime numbers. Thus, we represent the discussed here numbers as the vectors of exponents of the successive prime numbers. For the assumed restriction $n \leq 50$ we use only the first 15 prime numbers, that is 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47. Each discussed number is uniquely presented in the form of a 15-element table containing the exponents of the listed prime numbers. Hence, for multiplying two consecutive numbers we need only to add the respective exponents in the respective cells of the table. Whereas the division is performed by subtracting the exponents in the respective cells of the table. Only after executing all the multiplications and divisions, connected with computing the amount of numbers associated with the given multiset, we raise the prime numbers to the powers contained in the respective cells of the table and next the obtained results are mutually multiplied by receiving the final result.

The second applied algorithm is based on the preliminary selection of the n-digit numbers, the same as in the first algorithm, and next we operate on the obtained numbers by using the operator T_n k times (the value of k is arbitrary, depending on us, we usually took $k = 5$). From the obtained numbers we create a one-to-one sequence, on which we operate again by using the operator T_n, one time in this case, and next we create a one-to-one sequence from the T_n-images. We repeat this procedure as long as the number of elements in the T_n-image does change. Such created sequence forms the maximal invariant subset of the mapping T_n which is divided into the selective orbits. The corrected version of this algorithm separates before the fixed points of T_n, so the obtained maximal invariant subset is reduced by the set of all fixed points of T_n.

Let us explain the theoretical ground of the presented procedure.

Let X be a finite set and $F\colon X \to X$. We say that the set $Z \subseteq X$ the maximal invariant subset of the mapping F if $F(Z) = Z$ and for each $U \subseteq X$ if $Z \subseteq U$ and $F(U) = U$ then $U = Z$. Then the given below fundamental theorem holds.

Theorem 4.1. *The following conditions are equivalent:*
1) $X_{\mathrm{inv}} \subseteq X$ is the maximal invariant subset of the mapping F,
2) if $n \in \mathbb{N}$ and $F^{n+1}(X) = F^n(X)$, then $F^n(X) = X_{\mathrm{inv}}$,
3) if $F(Y) = Y$ and

$$\mathrm{card}\,\mathrm{Y} := \lim_{\mathrm{k}\to\infty} \frac{1}{\mathrm{k}} \sum_{\mathrm{l}=1}^{\mathrm{k}} \mathrm{card}\,\mathrm{F}^{\mathrm{l}}(\mathrm{X}), \tag{3}$$

then $Y = X_{\mathrm{inv}}$.

Remark 4.2. As a curiosity connected with formula (3) let us present one more formula, simply surprising for us,

$$\lim_{n\to\infty} \left(\lim_{k\to\infty} \frac{1}{k} \mathrm{card}\left(\mathrm{X_n} \cup \sigma(\mathrm{X_n}) \cup \sigma^2(\mathrm{X_n}) \cup \ldots \cup \sigma^{\mathrm{k}}(\mathrm{X_n})\right)\right)$$

where $X_n := \{1, 2, \ldots, n\}$ for every $n \in \mathbb{N}$, which determines the number of infinite orbits of the given permutation $\sigma\colon \mathbb{N} \to \mathbb{N}$! Proof of this formula can be provided by applying the decomposition of the permutation σ into cycles (T. Hudetz used it in his work [9] by suggesting that it was discovered by K. Thomsen [15]).

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